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# A note on shear flow past a sphere

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Flow of an incompressible inviscid fluid past a sphere is considered, where the flow upstream consists of a slight shear flow superimposed on a uniform stream. Secondary vorticity, produced by deformation of vortex elements as they are carried past the sphere, is determined by a method due to Lighthill (1956*b*). Components of vorticity are calculated from a drift function for which expressions were previously available only in part of the flow field. For the region in which no expansion is valid an exact integral expression is obtained to replace the rough numerical approximation used by Lighthill (1956*b*). The velocity distribution over the upstream part of the sphere is determined numerically using a Biot–Savart law. These results are required for the calibration of certain forms of Pitot tubes.

## 1. Introduction

Use of Pitot tubes in non-uniform flow fields has prompted a more detailed study of flow with slight shear past blunt objects. In particular Hall (1956) and Lighthill (1957 *b*) have investigated how the addition of a parallel shear flow to a uniform stream affects the measurement of pressure. They found that the measured value is greater than that on the streamline which approaches along the axis of the tube; it is equal to the pressure on a streamline displaced by a certain amount in the direction of higher velocities. The manner in which this displacement effect modifies measurements of speed is discussed in the papers cited above.

In recent years Pitot tubes have been developed to measure direction of flow as well as speed. One such instrument, the Warden tube, consists of a hemisphere mounted on a cylindrical shaft. Pressure readings can be taken from five holes located on the hemisphere; speed and direction of flow are then obtained either from experimental calibration or by calculation. However, this calibration or calculation is commonly done assuming uniform flow conditions. This will lead to wrong deductions being made from measured pressures when the instrument is used in a shear flow such as a wake or a boundary layer, and therefore a more detailed analysis is necessary

The displacement of the stagnation streamline and the velocity field in the neighbourhood of the upstream stagnation points have been determined in the work already cited. For instruments like the Warden tube, however, knowledge of the velocity field is required over a larger region of the head of the tube. In

§ 2 the method developed by Lighthill (1956*b*) is used to determine velocities over most of the hemispherical head; an exact expression is derived to replace a rough numerical approximation used previously. Numerical results are given in § 3, and their application to the calibration of Warden tubes is discussed in some detail by Cousins (1969).

## 2. Calculation of the velocity field

We assume that it is adequate to represent the tube by a sphere and, using Cartesian co-ordinates $x$, $y$, $z$, we define a flow with constant shear by a velocity

$$\mathbf{v} = (U + Ay, 0, 0), \tag{1}$$

where $U$ and $A$ are constant. No exact solution to the flow past a sphere with this condition upstream has been found, but since the shear parameter $Aa/U$, where $a$ is the radius of the sphere, is small in most practical applications, we obtain a first approximation

$$\mathbf{v} = \mathbf{v}_0 + (Aa/U)\,\mathbf{v}_1 \tag{2}$$

to the velocity, where $\mathbf{v}_0$ is uniform incompressible flow past a sphere and $\mathbf{v}_1$ is derived from knowledge of vorticity $\mathbf{w}$ *via* a Biot–Savart law

$$\mathbf{v}_1(\mathbf{r}') = -\frac{1}{4\pi}\int_{r\geqslant a} \mathbf{w}(\mathbf{r})\,\Lambda\nabla\frac{1}{|\mathbf{r}'-\mathbf{r}|}\,dV, \tag{3}$$

the integration being performed over the volume outside the sphere.

As we only require details of the flow on the upstream part of the sphere, where the boundary layer is thin, we neglect viscosity and assume that the pressure on the sphere is given accurately by considering inviscid flow. Lighthill (1957*b*) shows that vorticity downstream of the sphere contributes little to the displacement effect. This supports the view that a Pitot tube, with a shaft downstream, can be represented adequately by a sphere, at least for the upstream effects. This conclusion is strengthened by the work of Wellicome (1967), who shows that experimental calibration of a Warden tube in conditions of uniform flow agrees closely with the theoretical calculation of uniform incompressible flow past a sphere.

We follow the approach developed by Lighthill (1956*a*, *b*). To avoid a divergent integral the vorticity is split into two parts; its value far upstream and the change from that value as vortex elements are carried past the sphere. Further terms must be added to (3) to ensure that the flow remains incompressible and the boundary condition on the sphere is satisfied. These extra terms may be derived from an image vorticity system. To determine vorticity we introduce a drift function $t$, defined by integrating

$$dt = \frac{dr}{v_r} = \frac{r\,d\theta}{v_\theta} \quad \text{with} \quad t - \frac{x}{U} \to 0 \quad \text{as} \quad x \to \infty \tag{4}$$

along a streamline of the primary flow $\mathbf{v}_0$, where $v_r$ and $v_\theta$ are velocity components of the primary flow with respect to spherical polar co-ordinates $r, \theta, \lambda$. Components of vorticity are then obtained from $U\,\partial t/\partial\rho_0$, where

$$\rho_0^2 = r^2 \sin^2\theta(1 - a^3/r^3) \tag{5}$$

defines streamlines of the primary flow for $\rho_0 =$ constant. Various asymptotic expansions for $U\,\partial t/\partial\rho_0$ have been obtained in the references cited. There remains, however, a large intermediate region of the flow field for which no such expansion is valid. We now derive an exact expression to replace the rough numerical approximation used previously. To avoid a divergent integral we write (4) in the form

$$U\,dt = \left\{-\rho_0 \operatorname{cosec}^2\theta + \left(\rho_0 \operatorname{cosec}^2\theta - \frac{r\operatorname{cosec}\theta}{1+a^3/2r^3}\right)\right\}d\theta, \tag{6}$$

from which we obtain

$$Ut = \rho_0 \cot\theta - \int_\theta^\pi \left(\rho_0 \operatorname{cosec}^2\theta - \frac{r\operatorname{cosec}\theta}{1+a^3/2r^3}\right)d\theta, \tag{7}$$

the integration being performed along a streamline of the primary flow. Differentiating with respect to $\rho_0$ keeping $\theta$ constant we obtain

$$U\frac{\partial t}{\partial\rho_0} = \cot\theta - \int_r^\infty \left\{\operatorname{cosec}^2\theta - \operatorname{cosec}\theta\,\frac{r}{\rho_0}\,\frac{1-a^3/r^3}{(1+a^3/2r^3)^3}(1+2a^3/r^3)\right\} \times \left[\frac{-\rho_0(1+a^3/2r^3)}{r(1-a^3/r^3)\,(r^2-a^3/r-\rho_0^2)^{\frac{1}{2}}}\right]dr, \tag{8}$$

where the expression in square brackets is $\partial\theta/\partial r$ with $\rho_0$ constant for $\theta > \frac{1}{2}\pi$. If we now substitute $k = a/r$, (8) becomes

$$U\frac{\partial t(\rho_0,\theta)}{\partial\rho_0} = \cot\theta + \frac{a}{\rho_0}\int_0^{a/r} \frac{1+k^3/2}{k^2(1-k^3-k^2\rho_0^2/a^2)^{\frac{1}{2}}}\left\{1-\frac{(1-k^3)^{\frac{1}{2}}\,(1+2k^3)}{(1+k^3/2)^3}\right\}dk, \tag{9}$$

where $\rho_0$ is constant during the integration. For $\theta < \frac{1}{2}\pi$ values of $U\,\partial t/\partial\rho_0$ are obtained from its values for $\theta > \frac{1}{2}\pi$ through the relation

$$t(\rho_0,\theta) = 2t(\rho_0,\tfrac{1}{2}\pi) - t(\rho_0,\pi-\theta). \tag{10}$$

## 3. Numerical results of the secondary velocity calculation

Components of secondary velocity on the surface of the sphere were obtained using a digital computer; a five-point Newton–Cotes formula was used in the integration. The computer program guarantees an accuracy to one decimal place with the tolerance used in this calculation, though the program is sufficiently precise for the next figure to be reliable unless the integrand is particularly badly behaved. The results displayed in table 1 are therefore given to two decimal places. The boundary condition on the sphere, that the secondary velocity has a zero radial component, is satisfied to an accuracy of 0·01, as are the symmetries

$$\begin{aligned} v_{1x}(\lambda) &= v_{1x}(-\lambda) = -v_{1x}(180-\lambda) = -v_{1x}(\lambda-180),\\ v_{1y}(\lambda) &= v_{1y}(-\lambda) = v_{1y}(180-\lambda) = v_{1y}(\lambda-180),\\ v_{1z}(\lambda) &= -v_{1z}(-\lambda) = v_{1z}(180-\lambda) = -v_{1z}(\lambda-180), \end{aligned}$$

where $\lambda$ is expressed in degrees. The second decimal place is therefore unlikely to be in greater error than one digit. In any case the values obtained for $\mathbf{v}_1$ have to be multiplied by the small parameter $Aa/U$.

| $\theta$ | $\lambda$ | 0 | 15 | 30 | 45 | 60 | 75 | 90 | 105 | 120 | 135 | 150 | 165 | 180 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 120 | $v_{1x}$ | −0·04 | −0·04 | −0·03 | −0·02 | −0·02 | −0·01 | 0 | 0·01 | 0·02 | 0·02 | 0·03 | 0·04 | 0·04 |
| | $v_{1y}$ | −0·02 | −0·05 | −0·08 | −0·16 | −0·25 | −0·35 | −0·45 | −0·35 | −0·25 | −0·16 | −0·08 | −0·05 | −0·02 |
| | $v_{1z}$ | 0 | 0·06 | 0·09 | 0·13 | 0·12 | 0·08 | 0 | −0·08 | −0·12 | −0·13 | −0·09 | −0·06 | 0 |
| 130 | $v_{1x}$ | −0·15 | −0·14 | −0·11 | −0·09 | −0·07 | −0·04 | 0 | 0·04 | 0·07 | 0·09 | 0·11 | 0·14 | 0·15 |
| | $v_{1y}$ | −0·12 | −0·14 | −0·17 | −0·26 | −0·37 | −0·44 | −0·52 | −0·44 | −0·37 | −0·26 | −0·17 | −0·14 | −0·12 |
| | $v_{1z}$ | 0 | 0·07 | 0·11 | 0·14 | 0·14 | 0·09 | 0 | −0·09 | −0·14 | −0·14 | −0·11 | −0·07 | 0 |
| 140 | $v_{1x}$ | −0·28 | −0·26 | −0·22 | −0·16 | −0·10 | −0·05 | 0 | 0·05 | 0·10 | 0·16 | 0·22 | 0·26 | 0·28 |
| | $v_{1y}$ | −0·34 | −0·36 | −0·39 | −0·44 | −0·49 | −0·56 | −0·63 | −0·56 | −0·49 | −0·44 | −0·39 | −0·36 | −0·34 |
| | $v_{1z}$ | 0 | 0·08 | 0·14 | 0·15 | 0·15 | 0·10 | 0 | −0·08 | −0·14 | −0·15 | −0·15 | −0·10 | 0 |
| 150 | $v_{1x}$ | −0·37 | −0·35 | −0·28 | −0·21 | −0·14 | −0·07 | 0 | 0·07 | 0·14 | 0·21 | 0·28 | 0·35 | 0·37 |
| | $v_{1y}$ | −0·64 | −0·66 | −0·67 | −0·69 | −0·71 | −0·74 | −0·77 | −0·74 | −0·71 | −0·69 | −0·67 | −0·66 | −0·64 |
| | $v_{1z}$ | 0 | 0·12 | 0·17 | 0·16 | 0·14 | 0·08 | 0 | −0·08 | −0·14 | −0·16 | −0·18 | −0·12 | 0 |
| 160 | $v_{1x}$ | −0·30 | −0·29 | −0·25 | −0·19 | −0·12 | −0·06 | 0 | 0·06 | 0·12 | 0·19 | 0·25 | 0·29 | 0·30 |
| | $v_{1y}$ | −0·87 | −0·87 | −0·88 | −0·88 | −0·88 | −0·89 | −0·89 | −0·89 | −0·88 | −0·88 | −0·88 | −0·87 | −0·87 |
| | $v_{1z}$ | 0 | 0·08 | 0·11 | 0·12 | 0·11 | 0·07 | 0 | −0·07 | −0·11 | −0·12 | −0·11 | −0·08 | 0 |
| 170 | $v_{1x}$ | −0·17 | −0·16 | −0·13 | −0·10 | −0·06 | −0·03 | 0 | 0·03 | 0·06 | 0·10 | 0·13 | 0·16 | 0·17 |
| | $v_{1y}$ | −0·97 | −0·97 | −0·97 | −0·97 | −0·97 | −0·97 | −0·97 | −0·97 | −0·97 | −0·97 | −0·97 | −0·97 | −0·97 |
| | $v_{1z}$ | 0 | 0·01 | 0·02 | 0·02 | 0·02 | 0·01 | 0 | −0·01 | −0·02 | −0·02 | −0·02 | −0·01 | −0 |
| 180 | $v_{1x}$ | | | | | | | 0 | | | | | | |
| | $v_{1y}$ | | | | | | | −0·97 | | | | | | |
| | $v_{1z}$ | | | | | | | 0 | | | | | | |

TABLE 1. Values of the secondary velocity at points on the surface of the sphere. For $-180 < \lambda < 0$, $v_{1x}(\lambda) = v_{1x}(-\lambda)$, $v_{1y}(\lambda) = v_{1y}(-\lambda)$, $v_{1z}(\lambda) = -v_{1z}(-\lambda)$.

The limits of integration were varied and it was found that the contribution to the velocity from field points at a greater distance than $50a$ from the centre of the sphere was negligible, so that the upper limit for integration with respect to $r$ was finally taken as $50a$. It was also confirmed that downstream vorticity contributed only a small fraction to the velocity, a result which was obtained by Lighthill (1957*b*). The value of $v_{1y}$ at $(-a, 0, 0)$ is $-0{\cdot}97$, and the displacement of the upstream stagnation point on the sphere is

$$\lim_{r\to a} (r \sin\theta) = 0{\cdot}65\, Aa^2/U, \tag{11}$$

in agreement with Lighthill. It may be seen from table 1 that the maximum value of $|\mathbf{v}_1|$ is approximately unity, and, as components of $\mathbf{v}_1$ have to be multiplied by the shear parameter, the results are consistent with a first-order solution in $Aa/U$.

Practical use of these results for correcting readings obtained from a Pitot tube is discussed by Cousins (1969). Typical graphs of vorticity components are also given there.

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